A sphere of spherical objects

Jt w/ Deopurkar & Licata, Piland, WIP w/ Alcántara, Bin, Crepeau, Lack, Piland, Santos

<u>Aim</u>: Study the Harder-Narasimhan factors of spherical objects in 2CY categories

* A triangulated category $\mathcal{C}$ is 2CY if $\mathrm{Hom}^i(X,Y) \simeq \mathrm{Hom}^i(Y, X[2])^\vee$
* An object $X \in \mathcal{C}$ is <u>spherical</u> if $\mathrm{Hom}^i(X,X) \simeq H^*(S^2)$.

~~Prop [Mukai, Bridgeland, Huybrechts-Macri-Stellari]:~~

* We study Bridgeland stability conditions on these $\mathcal{C}$
* We only need the following fact: A BSC τ gives rise to a unique Harder-Narasimhan filtration

$$\begin{tikzcd} 0 = X_0 \arrow[r] & X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & X_n = X \arrow[d] \\ & A_1 \arrow[ul, dashed] & A_2 \arrow[ul, dashed] & & A_n \arrow[ul, dashed] \end{tikzcd}$$

The objects $A_1, \ldots, A_n$ are the <u>factors</u> - they appear in order of decreasing phase, and are τ-<u>semistable</u>.

<u>Prop</u> [Cor of Mukai, Huybrechts-Macri-Stellari, Bridgeland] : Let τ be a stability condition on a 2CY category $\mathcal{C}$ [+ technicalities)

Then its τ-HN factors satisfy $\mathrm{Hom}^1(A_i, A_i) = 0$.

$\Rightarrow$ stable subquotients of A_i are <u>spherical</u>.

<u>Thm</u> [B. Deopurkar-Licata] Let $\mathcal{C}$ be k-linear, Hom-finite, triangulated, admits a dg enhancement for some infinite field k. Let τ be a stability condition s.t every τ-semist spherical object is τ-stable. Then a spherical object X is determined uniquely by the ordered list $(A_1, \ldots, A_n)$ of its HN factors.

Our main example: the Khovanov-Seidel-Thomas category of a Coxeter graph Γ.

This has associated Coxeter gp W_Γ and associated ~~braid~~/Artin group B_Γ.

W_Γ has a geometric representation V_Γ with basis simple roots $\alpha_1, \dots, \alpha_n$.

$\mathcal{C}_\Gamma$ categorifies this rep and carries an action of B_Γ

$$B_\Gamma \circlearrowright \mathcal{C}_\Gamma \xrightarrow[\text{Grothendieck gp}]{\text{complexified}} V_\Gamma \circlearrowleft W_\Gamma$$

$$P_i \longmapsto \alpha_i$$

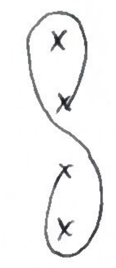
$$\mathrm{Hom}^\bullet(P_i, P_j) \xmapsto[\text{char}]{\text{Euler}} \langle \alpha_i, \alpha_j \rangle$$

* $\mathcal{C}_\Gamma$ is triangulated & 2CY
* P_i are spherical
* Generators σ_i of B_Γ act by spherical twists σ_{P_i}
* type A case has symplectic origin (Khovanov-Seidel, Thomas)
* non-simply-laced case requires care, studied by Heng-Licata, Dell-Heng-Licata.
* algebra that governs $\mathcal{C}_\Gamma$ is zig-zag algebra, subquotient of arc algebra from previous talk.
* ext closure $\langle P_1, \dots, P_n \rangle$ is the heart of a bdd t-structure called std t-structure

Back to (symplectic?) topology: consider type A_n situation.
[can ~~we~~ do similar things in other rk 2 cases + affine type A - WIP.]

Following Khovanov-Seidel, Thomas:

$\mathcal{C}_{E_n}$ vs $(n+1)$-punctured disk

spherical object $/[1]$ $\longleftrightarrow$ curve: 

P_i $/[1]$ $\longleftrightarrow$ 

Jordan-Hölder filtration factors + connecting maps $\longleftarrow$ curve

B_{n+1}-action $\longleftrightarrow$ B_{n+1}-action by half Dehn twists

$\boxed{Q}$ How to input ~~the~~ data of stability condition / HN factors?

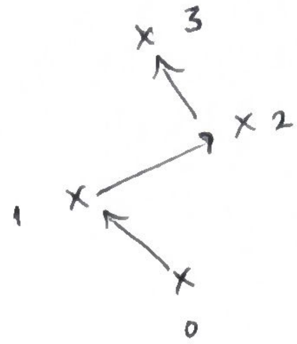
Thm [Thomas]: There exists a stability condition whose stable (spherical) objects are exactly  "left curves".

A stability condition is specified by ① heart $\heartsuit$ of bdd t-structure
② a stability fn $Z: K_0(\heartsuit) \to \mathbb{C}$; $\heartsuit \mapsto \mathbb{H}$.

* We already have $\heartsuit = \heartsuit_{std} = \langle P_1, \dots, P_n \rangle$
* Z specified by $\{Z(P_i)\}$: n elts of $\mathbb{H}$.

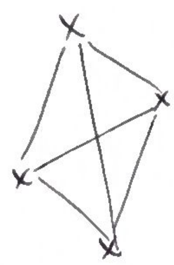
④

std stability condition $\longleftrightarrow$ some configuration



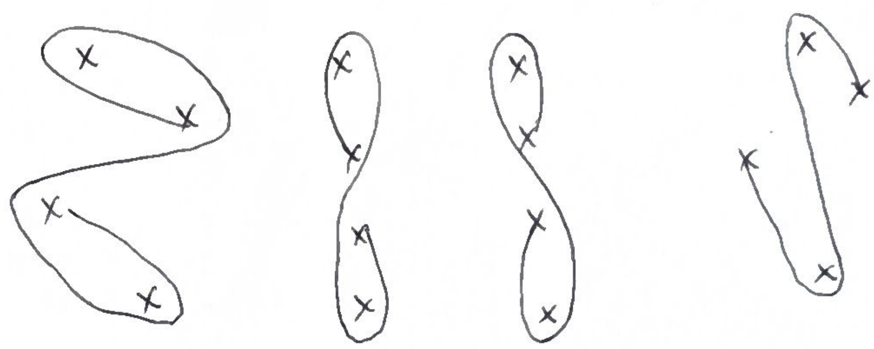
$Z(P_i)$ $\longleftrightarrow$ vector $i \to i{+}1$

spherical stables $\xleftrightarrow[\text{[BDL]}]{\text{Thm}}$ pseudo-straight segments



or

HN factors of spherical objects $\xleftrightarrow[\text{[BDL] [B-Piland]}]{\text{Thm}}$



HN factors + connecting maps $\xleftarrow[\text{[BDL]}]{\text{Thm}}$ curve

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<u>Combinatorics</u>

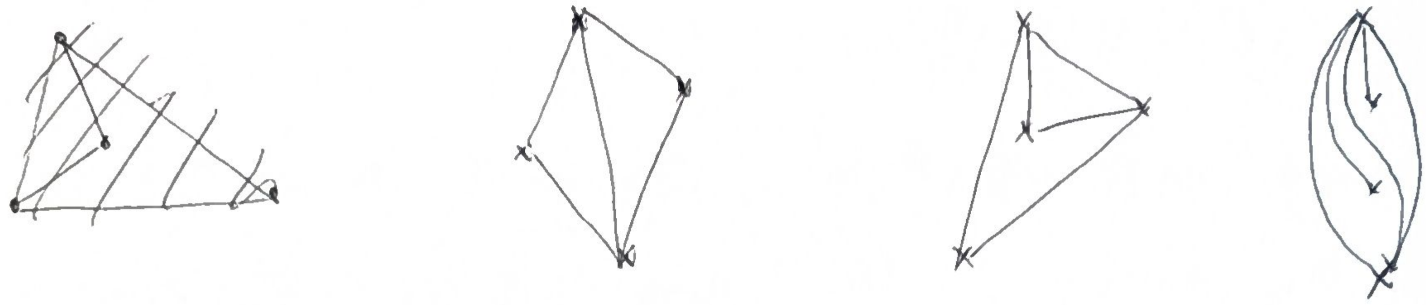
Consider a point configuration. A subset of pseudo-straight segments is called a pointed pseudo-triangulation or ppt if:

① No two segments cross

② ∀ pts ∃ half space that locally contains all segments

③ it is maximal w/ this property.

E.g.



Note : Every ppt contains all outer edges
If pt config is convex then ppt $\equiv$ triangulation.

We say a subset is a ppt* if it is a ppt minus one outside edge.

Observe : For any curve, its support $\in$ ppt*

• ppts studied for pts in general position by Rote - Santos - Streinu ['03]
• pts in linear position by B. - Piland [25]
• arbitrary position by [B.DL '25 + WIP ABBCLPS]

Facts : each ppt on $(n+1)$ pts has $(2n-1)$ edges
every internal edge can be uniquely exchanged.

Consider simplicial complex Σ whose max simplices $\equiv$ ppts, & Σ^* whose max simplices $\equiv$ ppt*s.

These are pure pseudomanifolds of dim $(2n-2)$ & $(2n-3)$ resp

Thm [BDL] : Σ is PL homeomorphic to ball of dim $2n-2$
$B_r \circlearrowright$ $\Sigma^* = \partial\Sigma$ and thus sphere of dim $2n-3$

X spherical $\hookrightarrow \partial\Sigma$; dense embedding
"sphere of sphericals"

6 Polytopal realisations

- convex configuration: associahedron (Loday ...)
- generic config: Rote-Santos-Streinu (expansive polytope)
- linear config: B.-Pilaud (wigglyhedron)
- arbitrary config: ~~B~~ ABBCLPS in progress

Further questions: other types ... ?