23/01/2024

Algorithms & computations w/ Bridgeland stability conditions

(Joint work w/ Anand Deopurkar & Anthony Licata)

* Central objects

Group $G \curvearrowright$ (suitably nice) (triangulated) category $\mathcal{C}$.

* Questions

Let $X \in \mathcal{C}$. As we apply elements of G, can we / how much can we:

(1) simplify X

(2) understand its growth complexity

(3) (algorithmically) build more complicated objects from a simpler X?

* We'll need:

- to find a $\mathcal{C}$.
- restrict the collection of X we focus on
- some measure of complexity on objects X.

* Aims

- Discuss the setup + rich measures of complexity provided by Bridgeland stability conditions
- Showcase some methods & results.

(2)

Bridgeland stability conditions (by analogy)

Let $\mathcal{C}$ be a triangulated category | bdd t-structures exist

[resp. abelian category, eg Rep Q or Coh X ;

resp vector space]

A Bridgeland stability condition [resp GIT stability condition]

specifies:

(1) A list of (semi) stable objects of $\mathcal{C}$ or heart of $\mathcal{C}$

(2) A mass $m_\tau(A) \in \mathbb{R}_{>0}$ for each semi-stable A

(3) A phase $\Phi_\tau(A) \in \mathbb{R}$ for each semi-stable A

~~compare~~ + compatibility conditions

[compare in GIT stability, $\exists$ slope $\forall$ stable object.

BSC is like a real cover?]

Key condition: existence of Harder-Narasimhan filtration.

Every $X \in \mathcal{C}$ has a unique filtration with semistable factors of strictly decreasing phase.

[compare every $v \in V$ has a unique expression in chosen basis]

Defn: If $X \in \mathcal{C}$ then $m_\tau(X)$ = sum of $m_\tau(A)$ for semistable factors in HN filtration of X.

Possible measures of complexity

(1)

(2) # HN factors in HN filtration

(3) Phase spread := difference between the max & min phases among the HN factors of X

...

* <u>A category with a (braid) group action</u>

We'll be interested in 2CY categories ($\exists$ duality)

Let $G \subseteq SL_2(\mathbb{C})$ be a finite subgroup

We have the Kleinian singularity $\mathbb{C}^2/G$,

its minimal resolution $Y \xrightarrow{f} \mathbb{C}^2/G$ &

associated ADE Dynkin diagram Γ.

$\mathcal{C} :=$ full subcategory of $D^b\mathrm{Coh}(Y)$ consisting of objects E supported on $f^{-1}(0)$, such that $Rf_*(E) = 0$

~~[illegible]~~

* Std heart of $\mathcal{C}$ is generated by one simple object for each vertex of Γ.

[Usually $\to$ more algebraic description using reps of a quiver coming from Γ]

<u>E.g</u> $\Gamma = A_3$ graph $\bullet - \bullet - \bullet$

~~$\mathcal{C}$~~ $\mathcal{C}$ is generated by P_1, P_2, P_3

* Each P_i is a <u>spherical object</u> : $\mathrm{Ext}^*(P_i) \simeq H^\bullet(S^2)$

* Each P_i gives an auto-equivalence (spherical twist – Seidel, Thomas)

$\sigma_{P_i} : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$

* ~~σ~~ σ_{P_i} satisfy braid relations

$\sigma_{P_i}\sigma_{P_j}\sigma_{P_i} \simeq \sigma_{P_j}\sigma_{P_i}\sigma_{P_j}$ if $i - j$

$\sigma_{P_i}\sigma_{P_j} \simeq \sigma_{P_j}\sigma_{P_i}$ if $i \not- j$

$\Bigg\}$ $B_\Gamma \curvearrowright \mathcal{C}$!

(B_4 in our example)

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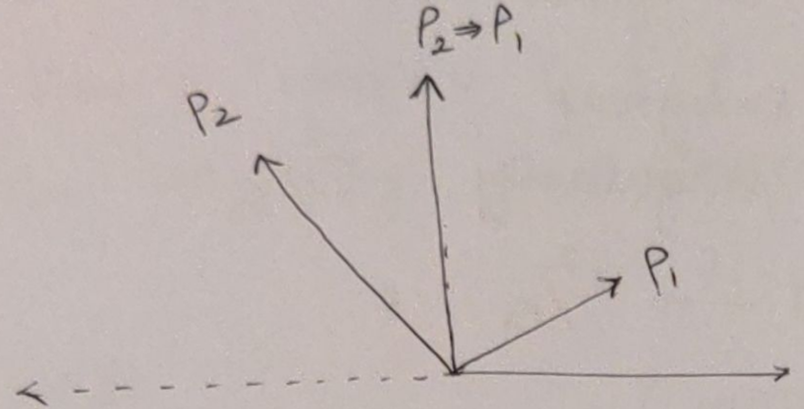
Fix a k-linear, hom-finite Δ category w/ fixed dg enhancement

Fix a stability condition (E.g. A_2)



Let X & Y be spherical objects of $\mathcal{C}$.

Consider $Y,\ \sigma_X Y,\ \sigma_X^2 Y,\ \sigma_X^3 Y, \dots$ Growth?

E.g. ① $Y = P_1$, $X = P_1$

$P_1,\ P_1[-1],\ P_1[-2], \dots$

② $Y = P_2$, $X = P_1$

$P_2,\quad P_1 \Rightarrow P_2,\quad P_1 \nearrow P_1 \Rightarrow P_2,\quad P_1 \nearrow P_1 \nearrow P_1 \Rightarrow P_2, \dots$

③ $Y = P_2 \Rightarrow P_1$, $X = P_1$

$P_2 \Rightarrow P_1,\quad P_2,\quad P_1 \Rightarrow P_2,\quad P_1 \nearrow P_1 \Rightarrow P_2, \dots$

$Y = P_1$, $X = P_2 \Rightarrow P_1$

$P_1,\quad P_2 \Rightarrow P_1 \nearrow P_1,\quad P_2 \Rightarrow P_1 \nearrow P_2 \Rightarrow P_1 \nearrow P_1, \dots$

Can measure $m_\tau(Y), m_\tau(\sigma_X Y), \dots$

Thm [BDL]

$$\lim_{n\to\infty} \frac{m_\tau(\sigma_X^n Y)}{n} = \begin{cases} m_\tau(X) \cdot \hom(X,Y) & \text{if } Y \neq X[i] \\ 0 & \text{otherwise} \end{cases}$$

Rmks

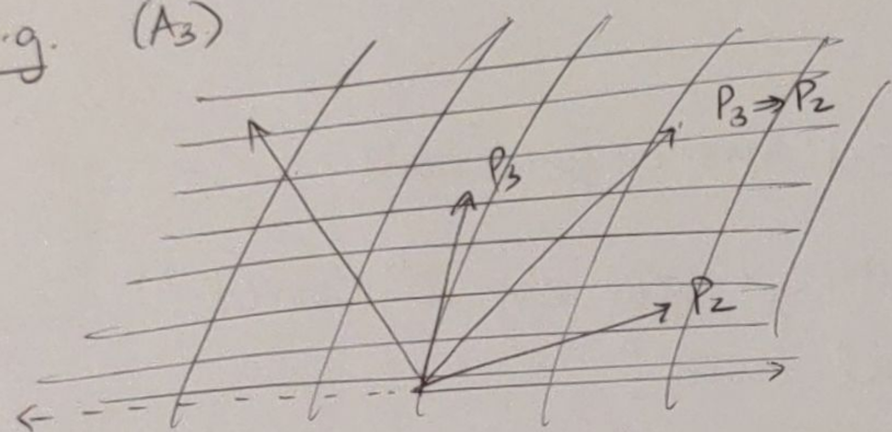
- Ties m to limit pts + compactification
- Growth of other gp elements ?!

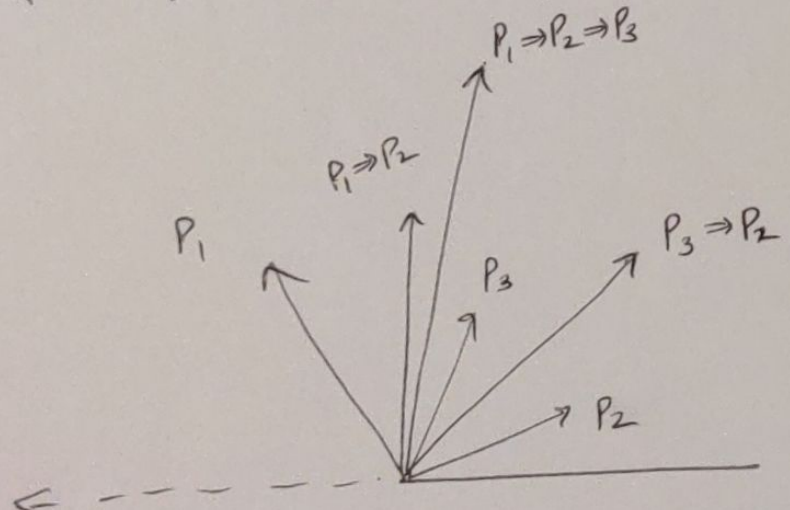
* Fact: If Y is spherical, then the HN filtration of Y also consists of spherical objects.

Q: How to simplify Y if Y not already stable?
Can we apply group elements to make Y stable?

Thm [BDL]: Let X be the factor of Y with top (resp bottom) HN phase.
Then the phase spread of $\sigma_X Y$ (resp. $\sigma_X^{-1} Y$) is smaller than the phase spread of Y.

E.g. (A_3)





$Y = P_2 \Rightarrow P_1, \; P_2 \Rightarrow P_3$ $X = P_2$

$\sigma_X^{-1} Y = (P_1 \Rightarrow P_2), \; P_3$ $Z = P_3$

$\sigma_Z^{-1} \sigma_X^{-1} Y = (P_1 \Rightarrow P_2)$ stable!

Rmks

(1) This algorithm terminates at a stable X — if it terminates at all!!

(2) Termination?

(3) If it terminates, gives several distinguished expressions

for $Y = \beta$ (semi-stable) $\to$ combinatorics +
group theory.

* Other ~~a~~ variations on the same theme

- Much more refined understanding e.g type A_2
  Y vs $\sigma_x Y$ have very different HN factors!
  subtle to understand how they change by gp action

Thm in A_2 : Finite automaton that governs HN evolution

- Construct semi-stables ~~to~~ w/ prescribed class in Grothendieck group
- Evolution of HN filtrations more generally?
- Other categories / other settings ~~ccd~~ eg $D^b\,Coh(K3)$
  - replacement of spherical twist?