ON THE LANGUAGE OF PALINDROMIC REDUCED EXPRESSIONS IN A COXETER GROUP

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1. Introduction

Let (W,S) be a Coxeter system. This means that S is a finite generating set of a group W consisting of elements of order two, with particular constraints on the relations (see Section 2). A *reflection* in W is defined to be any element that is conjugate to some element of S. Since elements of S each have order two, the inverse of any word in these generators is simply its reverse. Thus if a reflection t is written as a conjugate wsw^{-1} for some $s \in S$ and some $w \in W$, then t has a palindromic expression in the generating set S of W.

In fact, it is known (see, e.g. [3, Lemma 1.4] as well as [2, Exercise 1.10]) that any reflection has a palindromic *reduced* expression. Conversely, any word that has a palindromic reduced expression as wsw^{-1} is clearly a reflection.

The aim of this article is to answer a question of Biagioli–Hohlweg–Sasso [1], which asks whether the language of palindromic reduced expressions in W is regular. Certainly if W is a finite group, then this language is finite, and hence regular. The main theorem of [1] proves that the language of *reflection-prefixes* is regular for a general Coxeter system. However, we prove the following.

Theorem 1.1. *The language of palindromic reduced expressions in the Coxeter system of type affine A_2 is not regular. Thus the language of palindromic reduced expressions in a Coxeter system (W,S) is not regular in general.*

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2. Basic theory of Coxeter groups

We briefly recall the standard facts we need about Coxeter groups, omitting most proofs. We refer the reader to the standard reference books [2, 4] for more details and proofs.

Definition 2.1 (Coxeter system). A Coxeter system (W,S) consists of a group W and a distinguished finite generating set S. Furthermore, the relations of W are encoded by a function $m\colon S \times S \to \{2,\dots,\} \cup \{\infty\}$ with the property that $m(s,s) = 2$. More precisely, the only relations of W are $(ss')^{m(s,s')} = 1$ for any $s, s' \in S$ such that $m(s,s') \neq \infty$.

Henceforth in this section, fix a Coxeter system (W,S).

Definition 2.2 (Reflections). The elements of S are called the *simple reflections* of W. A *reflection* of W is any conjugate of an element of S. That is, the reflections are $\{wsw^{-1} \mid s \in S, w \in W\}$.

We note again that because elements of S have order two, every element $w \in W$ can simply be expressed as a product of elements of S. Indeed, the inverse of any product of elements of S is simply the reversed string.

Definition 2.3 (Length). Let $w \in W$. Then the *length* of w, denoted $\ell(w)$, is the minimal number of generators from S required to express w as a product $w = s_{i_1} \cdots s_{i_n}$.

The length function has a number of useful and pleasing properties, whose discussion we omit here. The Coxeter system (W,S) has a distinguished real representation V_W called the *standard geometric representation*, defined as follows. The underlying vector space of V_W is the free real vector space generated by elements $\{\alpha_s \mid s \in S\}$. Fix a bilinear form on V_W, defined on the basis as

$$(\alpha_s, \alpha_{s'}) = -\cos\left(\frac{\pi}{m(s,s')}\right), \tag{1}$$

where we set $\cos(\pi/\infty) = 1$. Now the action of a generator $s \in S \subset W$ on an element $v \in V_W$ is defined as

$$s(v) = v - 2(\alpha_s, v)\alpha_s.$$

Proposition 2.4. *The linear maps $s\colon V_W \to V_W$ defined by $s(v) = v - 2(\alpha_s, v)\alpha_s$ extend to a faithful linear representation $W \to GL(V_W)$, called the standard geometric representation of (W, S).*

Definition 2.5 (Root)**.** The basis elements $\alpha_s \in V_W$ for $s \in S$ are called *simple roots*. More generally, a *root* is any element of V_W that can be expressed as $w(\alpha_s)$ for some $s \in S$ and some $w \in W$.

Definition 2.6 (Depth)**.** Let α be a root. Suppose that $w = s_{i_1} \cdots s_{i_n}$ is a minimal length word such that $\alpha = w(\alpha_j)$ for some simple root α_j. Then *depth* of α is defined to be the quantity $n+1$.

The following statement is [2, Lemma 4.6.2], and explains how the depth of a root changes when it is acted upon by a simple reflection.

Proposition 2.7. *Let $s \in S$ and $\alpha \neq \alpha_s$ be a positive root. Then*

$$\operatorname{depth}(s(\alpha)) = \begin{cases} \operatorname{depth}(\alpha) - 1, & (\alpha, \alpha_s) > 0, \\ \operatorname{depth}(\alpha), & (\alpha, \alpha_s) = 0, \\ \operatorname{depth}(\alpha) + 1, & (\alpha, \alpha_s) < 0. \end{cases}$$

We also have the following result.

Proposition 2.8. *Let α be a positive root of depth d. Then*

$$\ell(s_\alpha) = 2\operatorname{depth}(\alpha) - 1.$$

3. Regular languages and the Myhill–Nerode theorem

In this section we recall the definition of regular languages, as well as a criterion for detecting a regular language that we will crucially use for the proof of Theorem 1.1.

Definition 3.1. Let Σ be an alphabet. We say that a language $L \subset \Sigma^*$ is *regular* if there exists a deterministic finite-state automaton M such that the language of words accepted by M is exactly L.

We do not review the definition and properties of finite-state automata because we do not need anything further about them. We refer the reader to e.g. [5, Chapter 1] for these details.

There is an explicit and elegant necessary and sufficient criterion to check whether a given language is regular. Moreover, in case the language is regular, this criterion outputs a minimal deterministic finite automaton that recognises it. The criterion is called the *Myhill–Nerode theorem*, which we write below as Theorem 3.3.

We first need the definition of a certain equivalence relation on words, which depends on a given language.

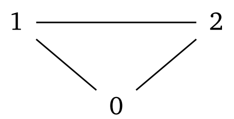
Definition 3.2. Let Σ be a finite alphabet, and let $L \subset \Sigma^*$ be any language. For $x, y \in \Sigma^*$, we say that $x \sim_L y$ if for every string $w \in \Sigma^*$, the string xw is in L if and only if the string yw is also in L.

We can now state the criterion, which is a fundamental result in the theory of regular languages. See, e.g. [5, Problem 1.52].

Theorem 3.3 (Myhill–Nerode theorem)**.** *Let Σ^* be an alphabet, and let $L \subset \Sigma^*$ be a language. Then L is regular if and only if the equivalence relation $\sim_L$ has finitely many equivalence classes on Σ^*. If L is regular, then there is a DFA M whose states are indexed by the equivalence classes of $\sim_L$, which recognises M. Furthermore, this is a DFA that recognises M that has the fewest possible states.*

4. Palindromic reduced expressions in affine type A_2

Consider the Coxeter system associated to the affine A_2 graph, drawn below.



We have the simple reflections $S = \{s_0, s_1, s_2\}$. For the remainder of this section, set

$$u_1 = s_0 s_1 s_2 \text{ and } u_2 = s_0 s_2 s_1,$$

where we remember the precise expressions of these group elements. Let δ be the so-called "imaginary root" $\alpha_0 + \alpha_1 + \alpha_2$. We first prove the following.

Lemma 4.1. *Let $i_1, \ldots, i_n$ be any elements of $\{1, 2\}$. We have $u_{i_1}^2 \cdots u_{i_n}^2(\alpha_0) = \alpha_0 + 3n\delta$. Furthermore,*

$$\operatorname{depth}(u_{i_1}^2 \cdots u_{i_n}^2(\alpha_0)) = \operatorname{depth}(\alpha_0 + 3n\delta) = 6n + 1.$$

Proof. We prove the first statement by induction on n. Suppose by induction that

$$u_{i_2}^2 \cdots u_{i_n}^2(\alpha_0) = \alpha_0 + 3(n-1)\delta.$$

Let us show that after multiplication on the left by $u_{i_1}^2$ for either value of i_1, the output is exactly $\alpha_0 + 3n\delta$.

We compute directly the effect of multiplying α_0 on the left by each letter of u_1^2 in turn, as follows:

$$\begin{aligned} \alpha_0 &\xrightarrow{s_2} (\alpha_0+\alpha_2) \xrightarrow{s_1} (\alpha_1+\delta) \xrightarrow{s_0} (\alpha_0+\alpha_1+\delta) \\ &\xrightarrow{s_2} (\alpha_2+2\delta) \xrightarrow{s_1} (\alpha_1+\alpha_2+2\delta) \xrightarrow{s_0} (\alpha_0+3\delta). \end{aligned} \tag{2}$$

Similarly, we compute directly the effect of multiplying α_0 on the left by each letter of u_2^2 in turn:

$$\begin{aligned} \alpha_0 &\xrightarrow{s_1} (\alpha_0+\alpha_1) \xrightarrow{s_2} (\alpha_2+\delta) \xrightarrow{s_0} (\alpha_0+\alpha_2+\delta) \\ &\xrightarrow{s_1} (\alpha_1+2\delta) \xrightarrow{s_2} (\alpha_1+\alpha_2+2\delta) \xrightarrow{s_0} (\alpha_0+3\delta). \end{aligned} \tag{3}$$

Recall also that $s_j(\delta) = \delta$ for any j. Thus we have, as desired,

$$u_1^2(\alpha_0 + 3(n-1)\delta) = u_2^2(\alpha_0 + 3(n-1)\delta) = \alpha_0 + 3n\delta.$$

To compute the depth, we use Proposition 2.7 to assert that the depth increases by one after applying each letter to α_0. At each step, we apply some letter s_j to some intermediate root α. We just need to check that the pairing (with respect to (1)) of the corresponding root α_j with α is negative at each step.

Up to some additional summands that are multiples of δ, each step of the calculation is one of the steps of either (2) or (3). Since the pairing of δ with any α_j is zero, the extra summands do not affect the pairings at each step. Now is easy to see case-by-case that at each step of (2) and (3) where we apply a letter s_j to a root α, the pairing between the corresponding α_j and α is negative. Thus the depth increases by one at every step, and the final depth is precisely $6n+1$ as desired. ☐

Corollary 4.2. *Let $i_1, \ldots, i_n$ be any elements of $\{1,2\}$. Then*

$$u_{i_1}^2 u_{i_2}^2 \cdots u_{i_n}^2 s_0 u_{i_n}^{-2} u_{i_{n-1}}^{-2} \cdots u_{i_1}^{-2} \tag{4}$$

is a palindromic reduced expression.

Proof. Observe that (4) is a palindromic writing for the reflection in the root $u_{i_1}^2 \cdots u_{i_n}^2(\alpha_0)$ consisting of exactly $12n+1$ letters. It follows by Lemma 4.1 and Proposition 2.8 that the expression (4) must be reduced. ☐

We also need the following two lemmas.

Lemma 4.3. *Let $i_1, \ldots, i_n$ be any elements of $\{1,2\}$. Let $w = u_{i_1}^2 u_{i_2}^2 \cdots u_{i_n}^2 s_0 u_{i_n}^{-2} u_{i_{n-1}}^{-2} \cdots u_{i_1}^{-2}$. Then there exists an expression w' such that ww' is also a palindromic reduced expression.*

Proof. Set $j_k = 2 - i_k$ for each $k \in [n]$. By expanding u_1 as $s_0 s_1 s_2$ and u_2 as $s_0 s_2 s_1$, we can re-parse the expression

$$w = u_{i_1}^2 u_{i_2}^2 \cdots u_{i_n}^2 s_0 u_{i_n}^{-2} u_{i_{n-1}}^{-2} \cdots u_{i_1}^{-2}$$

as

$$w = u_{i_1}^2 u_{i_2}^2 \cdots u_{i_n}^2 u_{j_n}^2 u_{j_{n-1}}^2 \cdots u_{j_1}^2 s_0.$$

Now set

$$w' = u_{j_1}^{-2} u_{j_{n-1}}^{-2} \cdots u_{j_n}^{-2} u_{i_n}^{-2} u_{i_{n-1}}^{-2} \cdots u_{i_1}^{-2}.$$

Then by Corollary 4.2, the expression ww' is a palindromic reduced expression. ☐

Note that the expression for w' we found in Lemma 4.3 is necessarily reduced, because it is a suffix of ww'. Furthermore, it has length exactly $12n$.

Lemma 4.4. *There exists a tuple $(i_1, \ldots, i_n)$ consisting of elements of $\{1,2\}$ with the following property. Let $w = u_{i_1}^2 u_{i_2}^2 \cdots u_{i_n}^2 s_0 u_{i_n}^{-2} u_{i_{n-1}}^{-2} \cdots u_{i_1}^{-2}$. Let w' be any expression such that ww' is a palindromic reduced expression. Then the length of w' is at least n.*

Proof. Consider a word w as above corresponding to some tuple $(i_1, \ldots, i_n)$. Suppose that there is a word w' of length $m < n$ that satisfies the desired property. Setting $x = u_{i_1}^2 u_{i_2}^2 \cdots u_{i_n}^2$, write

$$x = b_1 b_2 \cdots b_N,$$

where b_1 is a block consisting of the first m letters of x, then b_2 is a block consisting of the next m letters and so on, until b_{N-1}. Finally, b_N is the "remainder block", consisting of at most m letters, which is the remainder value of $6n$ modulo m. Written in this fashion, we see that

$$w = b_1 \cdots b_N s_0 \overline{b_N} \cdots \overline{b_1},$$

where $\overline{(\cdot)}$ denotes string reversal. Furthermore,

$$ww' = b_1 \cdots b_N s_0 \overline{b_N} \cdots \overline{b_1} w'$$

is also assumed to be palindromic. "Cancelling" successive blocks of length m from the front and back of ww', we observe that we have $b_1 = \overline{w'}$, $b_2 = b_1$, $b_3 = b_2$ and so on until $b_{N-1} = b_{N-2}$. Finally, b_N consists of the first $\ell(b_N)$ letters of b_{N-1}. The above calculation shows that if ww' is palindromic with $\ell(w') < n$, then the word x is periodic with some period $m < n$.

However, it is easy to construct a tuple $(i_1, \ldots, i_n)$ such that the word $x = u_{i_1}^2 \cdots u_{i_n}^2$ is *not* periodic for any period $m < n$. As a concrete example (although this is far from the only one), the sequence $(i_1, \ldots, i_n)$ taken as the prefix of length n of the infinite sequence $(1,2,1,2,2,1,2,2,2,1,2,2,2,2,\ldots)$ will produce such a non-periodic x. Thus the result is proved. □

Now we can prove the main theorem.

Proof of Theorem 1.1. Let L be the language of palindromic reduced expressions in the generating set $\{s_0, s_1, s_2\}$ of the Coxeter group of affine type A_2.

For each natural number n, there exists (by Lemmas 4.3 and 4.4) a palindromic reduced expression w_n with the following properties.

(1) There is a word w_n' of length $12n$ such that $w_n w_n'$ is in L.
(2) If w_n'' is any word such that $w_n w_n'' \in L$, then the length of w_n'' is at least n.

Fix a sequence (w_n) where each w_n satisfies the two properties above. Together, the two statements above imply that $w_n \not\sim_L w_m$ for $n > 12m$. In particular, there are infinitely many equivalence classes for $\sim_L$. The result now follows from Theorem 3.3. □

References

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